the total energy values for static and dynamic conditions are identical. If the velocity is increased, the impact values are considerably reduced. For further information, see Ref. 10.

10.6.6 Steady and Impulsive Vibratory Stresses

For steady vibratory stresses of a weight, W, supported by a beam or rod, the deflection of the bar, or beam, will be increased by the dynamic magnification factor. The relation is given by

$$\delta_{\text{dynamic}} = \delta_{\text{static}} \times \text{dynamic magnification factor}$$

An example of the calculating procedure for the case of no damping losses is

$$\delta_{\text{dynamic}} = \delta_{\text{static}} \times \frac{1}{1 - (\omega/\omega_n)^2} \tag{10.65}$$

where ω is the frequency of oscillation of the load and ω_n is the natural frequency of oscillation of a weight on the bar.

For the same beam excited by a single sine pulse of magnitude A in./sec^2 and a sec duration, then for $t < a$ a good approximation is

$$\sigma_{\text{dynamic}} = \frac{\delta_{\text{static}}(A/g)}{1 - \left(\dfrac{\omega}{4\pi\omega_n}\right)^2} \left[\sin \omega t - \frac{1}{4\pi^2}\left(\frac{\omega}{\omega_n}\right) \sin \omega_n t\right] \tag{10.66}$$

where A/g is the number of g's and ω is π/a.

10.7 SHAFTS, BENDING, AND TORSION

10.7.1 Definitions

Torsional stress. A bar is under torsional stress when it is held fast at one end, and a force acts at the other end to twist the bar. In a round bar (Fig. 10.23) with a constant force acting, the straight line *ab* becomes the helix *ad*, and a radial line in the cross section, *ob*, moves to the position *od*. The angle *bad* remains constant while the angle *bod* increases with the length of the bar. Each cross section of the bar tends to shear off the one adjacent to it, and in any cross section the shearing stress at any point is normal to a radial line drawn through the point. Within the shearing proportional limit, a radial line of the cross section remains straight after the twisting force has been applied, and the unit shearing stress at any point is proportional to its distance from the axis.

Twisting moment, T, is equal to the product of the resultant, P, of the twisting forces, multiplied by its distance from the axis, p.

Resisting moment, T_r, in torsion, is equal to the sum of the moments of the unit shearing stresses acting along a cross section with respect to the axis of the bar. If dA is an elementary area of the section at a distance of z units from the axis of a circular shaft (Fig. 10.23*b*), and c is the distance from the axis to the outside of the cross section where the unit shearing stress is τ, then the unit shearing stress acting on dA is $(\tau z/c)\, dA$, its moment with respect to the axis is $(\tau z^2/c)\, dA$, an the sum of all the moments of the unit shearing stresses on the cross section is $\int (\tau z^2/c)\, dA$. In

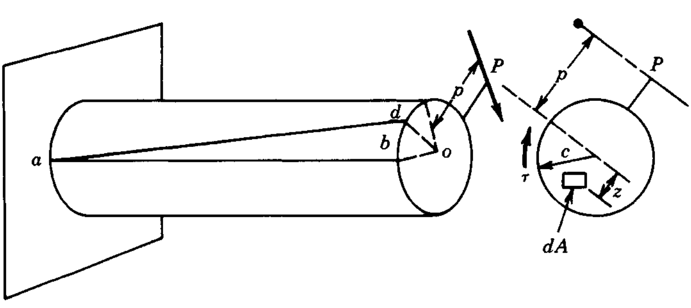


Fig. 10.23 Round bar subject to torsional stress.

this expression the factor $\int z^2\, dA$ is the polar moment of inertia of the section with respect to the axis. Denoting this by J, the resisting moment may be written $\tau J/c$.

The polar moment of inertia of a surface about an axis through its center of gravity and perpendicular to the surface is the sum of the products obtained by multiplying each elementary area by the square of its distance from the center of gravity of its surface; it is equal to the sum of the moments of inertia taken with respect to two axes in the plane of the surface at right angles to each other passing through the center of gravity. It is represented by J, inches^4. For the cross section of a round shaft,

$$J = \tfrac{1}{32}\pi d^4 \quad \text{or} \quad \tfrac{1}{2}\pi r^4 \tag{10.67}$$

For a hollow shaft,

$$J = \tfrac{1}{32}\pi(d^4 - d_1^4) \tag{10.68}$$

where d is the outside and d_1 is the inside diameter, inches, or

$$J = \tfrac{1}{2}\pi(r^4 - r_1^4) \tag{10.69}$$

where r is the outside and r_1 the inside radius, inches.

The polar radius of gyration, k_p, sometimes is used in formulas; it is defined as the radius of a circumference along which the entire area of a surface might be concentrated and have the same polar moment of inertia as the distributed area. For a solid circular section,

$$k_p^2 = \tfrac{1}{8}d^2 \tag{10.70}$$

For a hollow circular section,

$$k_p^2 = \tfrac{1}{8}(d^2 - d_1^2) \tag{10.71}$$

10.7.2 Determination of Torsional Stresses in Shafts

Torsion Formula for Round Shafts

The conditions of equilibrium require that the twisting moment, T, be opposed by an equal resisting moment, T_r, so that for the values of the maximum unit shearing stress, τ, within the proportional limit, the torsion formula for round shafts becomes

$$T_r = T = \tau \frac{J}{c} \tag{10.72}$$

if τ is in pounds per square inch, then T_r and T must be in pound-inches, J is in inches^4, and c is in inches. For solid round shafts having a diameter, d, inches,

$$J = \tfrac{1}{32}\pi d^4 \quad \text{and} \quad c = \tfrac{1}{2}d \tag{10.73}$$

and

$$T = \tfrac{1}{16}\pi d^3 \tau \quad \text{or} \quad \tau = \frac{16T}{\pi d^3} \tag{10.74}$$

For hollow round shafts,

$$J = \frac{\pi(d^4 - d_1^4)}{32} \quad \text{and} \quad c = \tfrac{1}{2}d \tag{10.75}$$

and the formula becomes

$$T = \frac{\tau\pi(d^4 - d_1^4)}{16d} \quad \text{or} \quad \tau = \frac{16Td}{\pi(d^4 - d_1^4)} \tag{10.76}$$

The torsion formula applies only to solid circular shafts or hollow circular shafts, and then only when the load is applied in a plane perpendicular to the axis of the shaft and when the shearing proportional limit of the material is not exceeded.

Shearing Stress in Terms of Horsepower

If the shaft is to be used for the transmission of power, the value of T, pound-inches, in the above formulas becomes $63{,}030H/N$, where H = horsepower to be transmitted and N = revolutions per minute. The maximum unit shearing stress, pounds per square inch, then is

$$\text{For solid round shafts:} \qquad \tau = \frac{321{,}000H}{Nd^3} \tag{10.77}$$

$$\text{For hollow round shafts:} \qquad \tau = \frac{321{,}000Hd}{N(d^4 - d_1^4)} \tag{10.78}$$

If τ is taken as the allowable unit shearing stress, the diameter, d, inches, necessary to transmit a given horsepower at a given shaft speed can then be determined. These formulas give the stress due to torsion only, and allowance must be made for any other loads, as the weight of shaft and pulley, and tension in belts.

Angle of Twist

When the unit shearing stress τ does not exceed the proportional limit, the angle *bod* (Fig. 10.23) for a solid round shaft may be computed from the formula

$$\theta = \frac{Tl}{GJ} \tag{10.79}$$

where θ = angle in radians; l = length of shaft in inches; G = shearing modulus of elasticity of the material; T = twisting moment, pound-inches. Values of G for different materials are steel, 12,000,000; wrought iron, 10,000,000; and cast iron, 6,000,000.

When the angle of twist on a section begins to increase in a greater ratio than the twisting moment, it may be assumed that the shearing stress on the outside of the section has reached the proportional limit. The shearing stress at this point may be determined by substituting the twisting moment at this instant in the torsion formula.

Torsion of Noncircular Cross Sections

The analysis of shearing stress distribution along noncircular cross sections of bars under torsion is complex. By drawing two lines at right angles through the center of gravity of a section before twisting, and observing the angular distortion after twisting, it as been found from many experiments that in noncircular sections the shearing unit stresses are not proportional to their distances from the axis. Thus in a rectangular bar there is no shearing stress at the corners of the sections, and the stress at the middle of the wide side is greater than at the middle of the narrow side. In an elliptical bar the shearing stress is greater along the flat side than at the round side.

It has been found by tests[5,11] as well as by mathematical analysis that the torsional resistance of a section, made up of a number of rectangular parts, is approximately equal to the sum of the resistances of the separate parts. It is on this basis that nearly all the formulas for noncircular sections have been developed. For example, the torsional resistance of an I-beam is approximately equal to the sum of the torsional resistances of the web and the outstanding flanges. In an I-beam in torsion the maximum shearing stress will occur at the middle of the side of the web, except where the flanges are thicker than the web, and then the maximum stress will be at the midpoint of the width of the flange. Reentrant angles, as those in I-beams and channels, are always a source of weakness in members subjected to torsion. Table 10.8 gives values of the maximum unit shearing stress τ and the angle of twist θ induced by twisting bars of various cross sections, it being assumed that τ is not greater than the proportional limit.

Torsion of thin-wall closed sections, Fig. 10.24,

$$T = 2qA \tag{10.80}$$

$$q = \tau t \tag{10.81}$$

$$\theta_i = \frac{\theta}{L} = \frac{T}{2A}\frac{1}{2AG}\frac{S}{t} = \frac{T}{GJ} \tag{10.82}$$

where S is the arc length around area A over which τ acts for a thin-wall section; shear buckling should be checked. When more than one cell is used[1,12] or if section is not constructed of a single material,[12] the calculations become more involved:

$$J = \frac{4A^2}{\oint ds/t} \tag{10.83}$$

Table 10.8 Formulas for Torsional Deformation and Stress

General formulas: $\theta = \frac{TL}{KG}$, $\tau = \frac{T}{Q}$, where θ = angle of twist, radians; T = twisting moment, in.-lb; L = length, in.; τ = unit shear stress, psi; G = modulus of rigidity, psi; K, in.^{4}; and Q, in.^{3} are functions of the cross section.

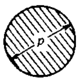
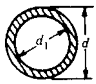
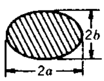
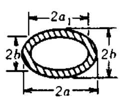
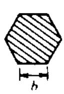
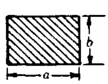
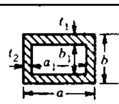
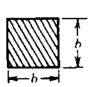
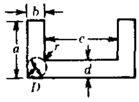
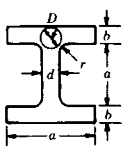
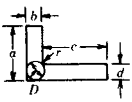
Shape	Formula for K in $\theta = \frac{TL}{KG}$	Formula for Shear Stress
	$K = \frac{\pi d^4}{32}$	$\tau = \frac{16T}{\pi d^3}$
	$K = 1/32\,\pi(d^4 - d_1^4)$	$\tau = \frac{16Td}{\pi(d^4 - d_1^4)}$
	$K = 2/3\,\pi r t^3$	$\tau = \frac{3T}{2\pi r t^2}$
	$K = \frac{\pi a^3 b^3}{a^2 + b^2}$	$\tau = \frac{2T}{\pi a b^2}$
	$K = \frac{\pi a_1^3 b_1^3}{a_1^2 + b_1^2}[(1+q)^4 - 1]$ $\quad q = \frac{a - a_1}{a_1}$ $\quad q = \frac{b - b_1}{b_1}$	$\tau = \frac{2T}{\pi a_1 b_1^2[(1+q)^4 - 1]}$
	$K = \frac{b^4\sqrt{3}}{80}$	$\tau = \frac{20T}{b^3}$
	$K = 2.69b^4$	$\tau = \frac{1.09T}{b^3}$
	$K = \frac{ab^3}{16}\left[\frac{16}{3} - 3.36\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$	$\tau = \frac{(3a + 1.8b)T}{a^2 b^2}$
	$K = \frac{2t_1 t_2(a - t_2)^2(b - t_1)^2}{at_2 + bt_1 - t_2^2 - t_1^2}$	$\tau = \frac{T}{2t_2(a - t_2)(b - t_1)}$
	$K = 0.1406b^4$	$\tau = \frac{4.8T}{b^3}$

Table 10.8 (*Continued*)

General formulas: $\theta = \dfrac{TL}{KG}$, $\tau = \dfrac{T}{Q}$, where θ = angle of twist, radians; T = twisting moment, in.-lb; L = length, in.; τ = unit shear stress, psi; G = modulus of rigidity, psi; K, in.^{4}; and Q, in.^{3} are functions of the cross section.

Shape	Formula for K in $\theta = \dfrac{TL}{KG}$	Formula for Shear Stress
	r = fillet radius D = diameter largest inscribed circle $K = 2K_1 + K_2 + 2\alpha D^4$ $K_1 = ab^3\left[\frac{1}{3} - 0.21\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$ $K_2 = cd^3\left[\frac{1}{3} - 0.105\frac{d}{c}\left(1 - \frac{d^4}{192c^4}\right)\right]$ $\alpha = \frac{b}{d}\left(0.07 + 0.076\frac{r}{b}\right)$	For all solid sections of irregular form the maximum shear stress occurs at or very near one of the points where the largest inscribed circle touches the boundary, and of these, at the one where the curvature of the boundary is algebraically least. (Convexity represents positive, concavity negative, curvature of the boundary.) At a point where the curvature is positive (boundary of section straight or convex) this maximum stress is given approximately by: $\tau = G\frac{\theta}{L}c$ or $\tau = \frac{T}{K}c$ where $c = \dfrac{D}{1 + \frac{\pi^2 D^4}{16A^2}} \times \left[1 + 0.15\left(\frac{\pi^2 D^4}{16A^2} - \frac{D}{2r}\right)\right]$ where D = diameter of largest inscribed circle, r = radius of curvature of boundary at the point (positive for this case), A = area of the section.
	$K = 2K_1 + K_2 + 2\alpha D^4$ $K_1 = ab^3\left[\frac{1}{3} - 0.21\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$ $K_2 = \frac{1}{3}cd^3$ $\alpha = \frac{t}{t_1}\left(0.15 + 0.1\frac{r}{b}\right)$ $t = b$ if $b < d$ $t = d$ if $d < b$ $t_1 = b$ if $b > d$ $t_1 = d$ if $d > b$	
	$K = K_1 + K_2 + \alpha D^4$ $K_1 = ab^3\left[\frac{1}{3} - 0.21\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$ $K_2 = cd^3\left[\frac{1}{3} - 0.105\frac{d}{c}\left(1 - \frac{d^4}{192c^4}\right)\right]$ $\alpha = \frac{b}{d}\left(0.07 + 0.076\frac{r}{b}\right)$	

Ultimate Strength in Torsion

In a torsion failure, the outer fibers of a section are the first to shear, and the rupture extends toward the axis as the twisting is continued. The torsion formula for round shafts has no theoretical basis after the shearing stresses on the outer fibers exceed the proportional limit, as the stresses along the section then are no longer proportional to their distances from the axis. It is convenient, however, to compare the torsional strength of various materials by using the formula to compute values of τ at which rupture takes place. These computed values of the maximum stress sustained before rupture are somewhat higher for iron and steel than the ultimate strength of the materials in direct shear. Computed values of the ultimate strength in torsion are found by experiment to be: cast iron, 30,000 psi; wrought iron, 55,000 psi; medium steel, 65,000 psi; timber, 2000 psi. These computed values of twisting strength may be used in the torsion formula to determine the probable twisting moment that will cause rupture of a given round bar or to determine the size of a bar that will be ruptured by a given twisting moment. In design, large factors of safety should be taken, especially when the stress is reversed as in reversing engines and when the torsional stress is combined with other stresses as in shafting.

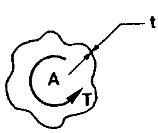


Fig. 10.24 Thin-walled tube.

10.7.3 Bending and Torsional Stresses

The stress for combined bending and torsion can be found from Eqs. (10.20), shear theory, and (10.22), distortion energy, with $\sigma_y = 0$:

$$\frac{\sigma_w}{2} = \sqrt{\left(\frac{Mc}{2I}\right)^2 + \left(\frac{Tr}{J}\right)^2} \tag{10.84}$$

For solid round rods, this equation reduces to

$$\frac{\sigma_w}{2} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2} \tag{10.85}$$

From distortion energy

$$\sigma = \sqrt{\left(\frac{Mc}{I}\right)^2 + 3\left(\frac{Tr}{J}\right)^2} \tag{10.86}$$

For solid round rods, the equation yields

$$\sigma = \frac{32}{\pi d^3}\sqrt{M^2 + \tfrac{3}{4}T^2} \tag{10.87}$$

10.8 COLUMNS

10.8.1 Definitions

A column or strut is a bar or structural member under axial compression, which has an unbraced length greater than about eight or ten times the least dimension of its cross section. On account of its length, it is impossible to hold a column in a straight line under a load; a slight sidewise bending always occurs, causing flexural stresses in addition to the compressive stresses induced directly by the load. The lateral deflection will be in a direction perpendicular to that axis of the cross section about which the moment of inertia is the least. Thus in Fig. 10.25*a* the column will bend in a direction perpendicular to *aa*, in Fig. 10.25*b* it will bend perpendicular to *aa* or *bb*, and in Fig. 10.25*c* it is likely to bend in any direction.

Radius of gyration of a section with respect to a given axis is equal to the square root of the quotient of the moment of inertia with respect to that axis, divided by the area of the section, that is

$$k = \sqrt{\frac{I}{A}}; \qquad \frac{I}{A} = k^2 \tag{10.88}$$

where I is the moment of inertia and A is the sectional area. Unless otherwise mentioned, an axis

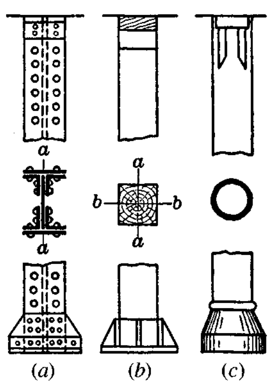


Fig. 10.25 Column end designs.

through the center of gravity of the section is the axis considered. As in beams, the moment of inertia is an important factor in the ability of the column to resist bending, but for purposes of computation it is more convenient to use the radius of gyration.

Length of a column is the distance between points unsupported against lateral deflection.

Slenderness ratio is the length l divided by the least radius of gyration k, both in inches. For steel, a *short column* is one in which $l/k < 20$ or 30, and its failure under load is due mainly to direct compression; in a *medium-length column*, $l/k =$ about 30–175, failure is by a combination of direct compression and bending; in a *long column*, $l/k >$ about 175–200, failure is mainly by bending. For timber columns these ratios are about 0–30, 30–90, and above 90 respectively. The load which will cause a column to fail decreases as l/k increases. The above ratios apply to round-end columns, If the ends are fixed (see below), the effective slenderness ratio is one-half that for round-end columns, as the distance between the points of inflection is one-half of the total length of the column. For flat ends it is intermediate between the two.

Conditions of ends. The various conditions which may exist at the ends of columns usually are divided into four classes: (1) Columns with round ends; the bearing at either end has perfect freedom of motion, as there would be with a ball-and-socket joint at each end. (2) Columns with hinged ends; they have perfect freedom of motion at the ends in one plane, as in compression members in bridge trusses where loads are transmitted through end pins. (3) Columns with flat ends; the bearing surface is normal to the axis of the column and of sufficient area to give at least partial fixity to the ends of the columns against lateral deflection. (4) Columns with fixed ends; the ends are rigidly secured, so that under any load the tangent to the elastic curve at the ends will be parallel to the axis in its original position.

Experiments prove that columns with fixed ends are stronger than columns with flat, hinged, or round ends, and that columns with round ends are weaker than any of the other types. Columns with hinged ends are equivalent to those with round ends in the plane in which they have free movement; columns with flat ends have a value intermediate between those with fixed ends and those with round ends. If often happens that columns have one end fixed and one end hinged, or some other combination. Their relative values may be taken as intermediate between those represented by the condition at either end. The extent to which strength is increased by fixing the ends depends on the length of column, fixed ends having a greater effect on long columns than on short ones.

10.8.2 Theory

There is no exact theoretical formula that gives the strength of a column of any length under an axial load. Formulas involving the use of empirical coefficients have been deduced, however, and they give results that are consistent with the results of tests.

Euler's Formula

Euler's formula assumes that the failure of a column is due solely to the stresses induced by sidewise bending. This assumption is not true for short columns, which fail mainly by direct compression, nor is it true for columns of medium length. The failure in such cases is by a combination of direct compression and bending. For columns in which $l/k > 200$, Euler's formula is approximately correct and agrees closely with the results of tests.

Let P = axial load, pounds; l = length of column, inches; I = least moment of inertia, inches^4; k = least radius of gyration, inches; E = modulus of elasticity; y = lateral deflection, inches, at any point along the column, that is caused by load P. If a column has round ends, so that the bending is not restrained, the equation of its elastic curve is

$$EI\frac{d^2y}{dx^2} = -Py \tag{10.89}$$

when the origin of the coordinate axes is at the top of the column, the positive direction of x being taken downward and the positive direction of y in the direction of the deflection. Integrating the above expression twice and determining the constants of integration give

$$P = \Omega\pi^2\frac{EI}{l^2} \tag{10.90}$$

which is Euler's formula for long columns. The factor Ω is a constant depending on the condition of the ends. For round ends $\Omega = 1$; for fixed ends $\Omega = 4$; for one end round and the other fixed $\Omega = 2.05$. P is the load at which, if a slight deflection is produced, the column will not return to its original position. If P is decreased, the column will approach its original position, but if P is increased, the deflection will increase until the column fails by bending.

For columns with value of l/k less than about 150, Euler's formula gives results distinctly higher than those observed in tests. Euler's formula is now little used except for long members and as a basis for the analysis of the stresses in some types of structural and machine parts. It always gives an *ultimate* and never an allowable load.

Secant Formula

The deflection of the column is used in the derivation of the Euler formula, but if the load were truly axial it would be impossible to compute the deflection. If the column is assumed to have an initial eccentricity of load of e in. (see Ref. 7, for suggested values of e), the equation for the deflection y becomes

$$y_{max} = e\left(\sec\frac{l}{2}\sqrt{\frac{P}{EI}} - 1\right) \tag{10.91}$$

The maximum unit compressive stress becomes

$$\sigma = \frac{P}{A}\left(1 + \frac{ec}{k^2}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right) \tag{10.92}$$

where l = length of column, inches; P = total load, pounds; A = area, square inches; I = moment of inertia, inches^4; k = radius of gyration, inches; c = distance from neutral axis to the most compressed fiber, inches; E = modulus of elasticity; both I and k are taken with respect to the axis about which bending takes place. The ASCE indicates $ec/k^2 = 0.25$ for central loading. Because the formula contains the secant of the angle $(l/2)\sqrt{P/EI}$, it is sometimes called the *secant formula.* It has been suggested by the Committee on Steel-Column Research[13,14] that the best rational column formula can be constructed on the secant type, although of course it must contain experimental constants.

The secant formula can be used also for columns that are eccentrically loaded, if e is taken as the actual eccentricity plus the assumed initial eccentricity.

Eccentric Loads on Short Compression Members

Where a direct push acting on a member does not pass through the centroid but at a distance e, inches, from it, both direct and bending stresses are produced. For short compression members in which column action may be neglected, the direct unit stress is P/A, where P = total load, pounds, and A = area of cross section, square inches. The bending unit stress is Mc/I, where $M = Pe$ = bending moment, pound-inches; c = distance, inches, from the centroid to the fiber in which the stress is desired; I = moment of inertia, inches^4. The total unit stress at any point in the section is $\sigma = P/A + Pec/I$, or $\sigma = (P/A)(1 + ec/k^2)$, since $I = Ak^2$, where k = radius of gyration, inches.

Eccentric Loads on Columns

Various column formulas must be modified when the loads are not balanced, that is, when the resultant of the loads is not in line with the axis of the column. If P = load, pounds, applied at a distance e in. from the axis, bending moment $M = Pe$. Maximum unit stress σ, pounds per square inch, due to this bending moment alone, is $\sigma = Mc/I = Pec/Ak^2$, where c = distance, inches, from the axis to the most remote fiber on the concave side; A = sectional area in square inches; k = radius of gyration in the direction of the bending, inches. This unit stress must be added to the unit stress that would be induced if the resultant load were applied in line with the axis of the column.

The secant formula, Eq. (10.92), also can be used for columns that are eccentrically loaded if e is taken as the actual eccentricity plus the assumed initial eccentricity.

Column Subjected to Transverse or Cross-Bending Loads

A compression member that is subjected to cross-bending loads may be considered to be (1) a beam subjected to end thrust or (2) a column subjected to cross-bending loads, depending on the relative magnitude of the end thrust and cross-bending loads, and on the dimensions of the member. The various column formulas may be modified so as to include the effect of cross-bending loads. In this form the modified secant formula for transverse loads is

$$\sigma = \frac{P}{A}\left[1 + (e + y)\frac{c}{k^2}\sec\frac{l}{2k}\sqrt{\frac{P}{AE}}\right] + \frac{Mc}{Ak^2} \tag{10.93}$$

In the formula, σ = maximum unit stress on concave side, pounds per square inch; P = axial end load, pounds; A = cross-sectional area, square inches; M = moment due to cross-bending load, pound-inches; y = deflection due to cross-bending load, inches; k = radius of gyration, inches; l = length of column, inches; e = assumed initial eccentricity, inches; c = distance, inches, from axis to the most remote fiber on the concave side.

10.8.3 Wooden Columns

Wooden Column Formulas

One of the principal formulas is that formerly used by the AREA, $P/A = \sigma_1(1 - l/60d)$, where P/A = allowable unit load, pounds per square inch; σ_1 = allowable unit stress in direct compression on short blocks, pounds per square inch; l = length, inches; d = least dimension, inches. This formula is being replaced rapidly by formulas recommended by the ASTM and AREA. Committees of these societies, working with the U.S. Forest Products Laboratory, classified timber columns in three groups (ASTM Standards, 1937, D245-37):

1. *Short Columns.* The ratio of unsupported length to least dimension does not exceed 11. For these columns, the allowable unit stress should not be greater than the values given in Table 10.9 under compression parallel to the grain.
2. *Intermediate-Length Columns.* Where the ratio of unsupported length to least dimension is greater than 10, Eq. (10.94), of the fourth power parabolic type, shall be used to determine allowable unit stress, until this allowable unit stress is equal to two-thirds of the allowable unit stress for short columns.

$$\frac{P}{A} = \sigma_1 \left[1 - \frac{1}{3}\left(\frac{l}{Kd}\right)^4\right] \tag{10.94}$$

   where P = total load, pounds; A = area, square inches; σ_1 = allowable unit compressive stress parallel to grain, pounds per square inch (see Table 10.9); l = unsupported length, inches; d = least dimension, inches; $K = l/d$ at the point of tangency of the parabolic and Euler curves, at which $P/A = \frac{2}{3}\sigma_1$. The value of K for any species and grade is $\pi/2\sqrt{E/6\sigma_1}$, where E = modulus of elasticity.
3. *Long Columns.* Where P/A as computed by Eq. (10.94) is less than $\frac{2}{3}\sigma_1$, Eq. (10.95) of the Euler type, which includes a factor of safety of 3, shall be used:

$$\frac{P}{A} = \frac{1}{36}\left[\frac{\pi^2 E}{(l/d)^2}\right] \tag{10.95}$$

Timber columns should be limited to a ratio of l/d equal to 50. No higher loads are allowed for square-ended columns. The strength of round columns may be considered the same as that of square columns of the same cross-sectional area.

Use of Timber Column Formulas

The values of E (modulus of elasticity) and σ_1 (compression parallel to grain) in the above formulas are given in Table 10.9. Table 10.10 gives the computed values of K for some common types of timbers. These may be substituted directly in Eq. (10.94) for intermediate-length columns, or may be used in conjunction with Table 10.11, which gives the strength of columns of intermediate length, expressed as a percentage of strength (σ_1) of short columns. In the tables, the term "continuously dry" refers to interior construction where there is no excessive dampness or humidity; "occasionally wet but quickly dry" refers to bridges, trestles, bleachers, and grandstands; "usually wet" refers to timber in contact with the earth or exposed to waves or tidewater.

10.8.4 Steel Columns

Types

Two general types of steel columns are in use: (1) rolled shapes and (2) built-up sections. The rolled shapes are easily fabricated, accessible for painting, neat in appearance where they are not covered, and convenient in making connections. A disadvantage is the probability that thick sections are of lower-strength material than thin sections because of the difficulty of adequately rolling the thick material. For the effect of thickness of material on yield point, see Ref. 14, p. 1377.

General Principles in Design

The design of steel columns is always a cut-and-try method, as no law governs the relation between area and radius of gyration of the section. A column of given area is selected, and the amount of load that it will carry is computed by the proper formula. If the allowable load so computed is less than that to be carried, a larger column is selected and the load for it is computed, the process being repeated until a proper section is found.

Table 10.9 Basic Stresses for Clear Material*

Species	Extreme Fiber in Bending or Tension Parallel to Grain	Maximum Horizontal Shear	Compression Perpendicular to Grain	Compression Parallel to Grain $L/d = 11$ or Less	Modulus of Elasticity in Bending
Softwoods					
Baldcypress (Southern cypress)	1900	150	300	1450	1,200,000
Cedars					
Redcedar, Western	1300	120	200	950	1,000,000
White-cedar, Atlantic (Southern white-cedar) and northern	1100	100	180	750	800,000
White-cedar, Port Orford	1600	130	250	1200	1,500,000
Yellow-cedar, Alaska (Alaska cedar)	1600	130	250	1050	1,200,000
Douglas-fir, coast region	2200	130	320	1450	1,600,000
Douglas-fir, coast region, close-grained	2350	130	340	1550	1,600,000
Douglas-fir, Rocky Mountain region	1600	120	280	1050	1,200,000
Douglas-fir, dense, all regions	2550	150	380	1700	1,600,000
Fir, California red, grand, noble, and white	1600	100	300	950	1,100,000
Fir, balsam	1300	100	150	950	1,000,000
Hemlock, Eastern	1600	100	300	950	1,100,000
Hemlock, Western (West Coast hemlock)	1900	110	300	1200	1,400,000
Larch, Western	2200	130	320	1450	1,500,000
Pine, Eastern white (Northern white), ponderosa, sugar, and Western white (Idaho white)	1300	120	250	1000	1,000,000
Pine, jack	1600	120	220	1050	1,100,000
Pine, lodgepole	1300	90	220	950	1,000,000
Pine, red (Norway pine)	1600	120	220	1050	1,200,000
Pine, southern yellow	2200	160	320	1450	1,600,000
Pine, southern yellow, dense	2550	190	380	1700	1,600,000
Redwood	1750	100	250	1350	1,200,000
Redwood, close-grained	1900	100	270	1450	1,200,000
Spruce, Engelmann	1100	100	180	800	800,000
Spruce, red, white, and Sitka	1600	120	250	1050	1,200,000
Tamarack	1750	140	300	1350	1,300,000
Hardwoods					
Ash, black	1450	130	300	850	1,100,000
Ash, commercial white	2050	185	500	1450	1,500,000
Beech, American	2200	185	500	1600	1,600,000
Birch, sweet and yellow	2200	185	500	1600	1,600,000
Cottonwood, Eastern	1100	90	150	800	1,000,000
Elm, American and slippery (white or soft elm)	1600	150	250	1050	1,200,000
Elm, rock	2200	185	500	1600	1,300,000
Gums, blackgum, sweetgum (red or sap gum)	1600	150	300	1050	1,200,000
Hickory, true and pecan	2800	205	600	2000	1,800,000
Maple, black and sugar (hard maple)	2200	185	500	1600	1,600,000
Oak, commercial red and white	2050	185	500	1350	1,500,000
Tupelo	1600	150	300	1050	1,200,000
Yellow poplar	1300	120	220	950	1,100,000

*These stresses are applicable with certain adjustments to material of any degree of seasoning.

(For use in determining working stresses according to the grade of timber and other applicable factors. All values are in pounds per square inch. U.S. Forest Products Laboratory.)

Table 10.10 Values of *K* for Columns of Intermediate Length

ASTM Standards, 1937, D245–37

	Continuously Dry		Occasionally Wet		Usually Wet	
Species	Select	Common	Select	Common	Select	Common
Cedar, western red	24.2	27.1	24.2	27.1	25.1	28.1
Cedar, Port Orford	23.4	26.2	24.6	27.4	25.6	28.7
Douglas fir, coast region	23.7	27.3	24.9	28.6	27.0	31.1
Douglas fir, dense	22.6	25.3	23.8	26.5	25.8	28.8
Douglas fir, Rocky Mountain region	24.8	27.8	24.8	27.8	26.5	29.7
Hemlock, west coast	25.3	28.3	25.3	28.3	26.8	30.0
Larch, western	22.0	24.6	23.1	25.8	25.8	28.8
Oak, red and white	24.8	27.8	26.1	29.3	27.7	31.1
Pine, southern	.....	27.3	.....	28.6	.....	31.1
Pine, dense	22.6	25.3	23.8	26.5	25.8	28.8
Redwood	22.2	24.8	23.4	26.1	25.6	28.6
Spruce, red, white, Sitka	24.8	27.8	25.6	28.7	27.5	30.8

A few general principles should guide in proportioning columns. The radius of gyration should be approximately the same in the two directions at right angles to each other; the slenderness ratio of the separate parts of the column should not be greater than that of the column as a whole; the different parts should be adequately connected in order that the column may function as a single unit; the material should be distributed as far as possible from the centerline in order to increase the radius of gyration.

Steel Column Formulas

A variety of steel column formulas are in use, differing mostly in the value of unit stress allowed with various values of l/k. See Ref. 15, for a summary of the formulas.

Test on Steel Columns

After the collapse of the Quebec Bridge in 1907 as a result of a column failure, the ASCE, the AREA, and the U.S. Bureau of Standards cooperated in tests of full-sized steel columns. The results of these tests are reported in Ref. 16, pp. 1583–1688. The tests showed that, for columns of the proportions commonly used, the effect of variation in the steel, kinks, initial stresses, and similar

Table 10.11 Strength of Columns of Intermediate Length, Expressed as a Percentage of Strength of Short Columns

ASTM Standards, 1937, D245–37
Values for expression $\{1 - \frac{1}{3}(l/Kd)^4\}$ in eq. 33

	Ratio of Length to Least Dimension in Rectangular Timbers, l/d																			
K	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
22	97	96	95	93	91	88	85	81	77	72	67	....								
23	98	97	95	94	92	90	87	84	81	77	72	67	....							
24	98	97	96	95	93	92	89	87	84	80	76	72	67	....						
25	98	98	97	96	94	93	91	89	86	83	80	76	72	67	....					
26	99	98	97	96	95	93	92	91	89	86	83	80	76	72	67	....				
27	99	98	98	97	96	95	93	92	90	88	85	82	79	74	71	67	....			
28	99	98	98	97	96	95	94	93	91	89	87	85	82	79	75	71	67	....		
29	99	99	98	98	97	96	95	94	92	91	89	87	84	82	79	75	71	67	......	
30	99	99	98	98	97	97	96	95	94	92	90	88	86	84	81	78	75	71	67	..
31	99	99	99	98	98	97	96	95	94	93	92	90	88	86	84	81	78	75	71	67

Note. This table can also be used for columns not rectangular, the l/d being equivalent to $0.289l/k$, where k is the least radius of gyration of the section.

defects in the column was more important than the effect of length. They also showed that the thin metal gave definitely higher strength, per unit area, than the thicker metal of the same type of section.

10.9 CYLINDERS, SPHERES, AND PLATES

10.9.1 Thin Cylinders and Spheres under Internal Pressure

A cylinder is regarded as thin when the thickness of the wall is small compared with the mean diameter, or $d/t > 20$. There are only tensile membrane stresses in the wall developed by the internal pressure p

$$\frac{\sigma_1}{R_1} + \frac{\sigma_2}{R_2} = \frac{p}{t} \tag{10.96}$$

In the case of a cylinder where R_1, the curvature, is R and R_2 is infinite, and the hoop stress is

$$\sigma_1 = \sigma_h = \frac{pR}{t} \tag{10.97}$$

If the two equations are compared, it is seen that the resistance to rupture by circumferential stress [Eq. (10.97)] is one-half the resistance to rupture by longitudinal stress [Eq. (10.98)]. For this reason cylindrical boilers are single riveted in the circumferential seams and double or triple riveted in the longitudinal seams.

From the equations of equilibrium, the longitudinal stress is

$$\sigma_2 = \sigma_L = \frac{pR}{2t} \tag{10.98}$$

For a sphere, using Eq. (10.96), $R_1 = R_2 = R$ and $\sigma_1 = \sigma_2$, making

$$\sigma_1 = \sigma_2 = \frac{pR}{2t} \tag{10.99}$$

In using the foregoing formulas to design cylindrical shells or piping, thickness t must be increased to compensate for rivet holes in the joints. Water pipes, particularly those of cast iron, require a high factor of safety, which results in increased thickness to provide security against shocks caused by water hammer or rough handling before they are laid. Equation (10.98) applies also to the stresses in the walls of a thin hollow sphere, hemisphere, or dome. When holes are cut, the tensile stresses must be found by the method used in riveted joints.

Thin Cylinders under External Pressure

Equations (10.97) and (10.98) apply equally well to cases of external pressure if P is given a negative sign, but the stresses so found are significant only if the pressure and dimensions are such that no buckling can occur.

10.9.2 Thick Cylinders and Spheres

Cylinders

When the thickness of the shell or wall is relatively large, as in guns, hydraulic machinery piping, and similar installations, the variation in stress from the inner surface to the outer surface is relatively large, and the ordinary formulas for thin wall cylinders are no longer applicable. In Fig. 10.26 the stresses, strains, and deflections are related[1,18,19] by

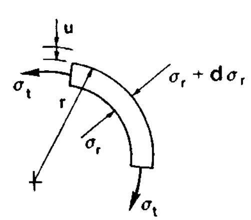


Fig. 10.26 Cylindrical element.

$$\sigma_t = \frac{E}{1-\nu^2}(\epsilon_t + \nu\epsilon_r) = \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu\frac{\partial u}{\partial r}\right] \tag{10.100}$$

$$\sigma_r = \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_t) = \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial r} + \nu\frac{u}{r}\right] \tag{10.101}$$

where E is the modulus and ν is Poisson's ratio. In a cylinder (Fig. 10.27) that has internal and external pressures, p_i and p_o; internal and external radii, a and b; $K = b/a$; the stresses are

$$\sigma_t = \frac{p_i}{K^2-1}\left(1 + \frac{b^2}{r^2}\right) - \frac{p_o K^2}{K^2-1}\left(1 + \frac{a^2}{r^2}\right) \tag{10.102}$$

$$\sigma_r = \frac{p_i}{K^2-1}\left(1 - \frac{b^2}{r^2}\right) - \frac{p_o K^2}{K^2-1}\left(1 - \frac{a^2}{r^2}\right) \tag{10.103}$$

if $p_o = 0$, and σ_t, σ_r are maximum at $r = a$; if $p_i = 0$, σ_t is maximum at $r = a$; and σ_r is maximum at $r = b$.

In shrinkage fits, Fig. 10.27, a hollow cylinder is pressed over a cylinder with a radial interference δ at $r = b$. p_f, the pressure between the cylinders, can be found from

$$\delta = \frac{bp_f}{E_o}\left(\frac{c^2+b^2}{c^2-b^2} + \nu_o\right) + \frac{bp_f}{E_i}\left(\frac{a^2+b^2}{b^2-a^2} - \nu_i\right) \tag{10.104}$$

The radial deflection can be found at a which shrinks and c which expands by knowing σ_r is zero and using Eqs. (10.100) and (10.101):

$$u_a = \frac{\sigma_t}{E_i}a, \qquad u_c = \frac{\sigma_t}{E_o}c \tag{10.105}$$

Spheres

The stress, strain, and deflections[19,20] are related by

$$\sigma_t = \frac{E}{1-\nu-2\nu^2}[\epsilon_t + \nu\epsilon_r] = \frac{E}{1-\nu-2\nu^2}\left[\frac{u}{r} + \nu\frac{\partial u}{\partial r}\right] \tag{10.106}$$

$$\sigma_r = \frac{E}{1-\nu-2\nu^2}[2\nu\epsilon_t + (1-\nu)\epsilon_r] = \frac{E}{1-\nu-2\nu^2}\left[2\nu\frac{u}{r} + (1-\nu)\frac{\partial u}{\partial r}\right] \tag{10.107}$$

The stresses for a thick wall sphere with internal and external pressure, p_i and p_o, and $K = b/a$ are

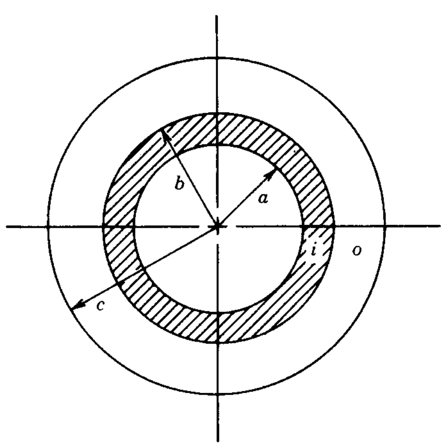


Fig. 10.27 Cylinder press fit.

$$\sigma_t = \frac{p_i(1 + b^3/2r^3)}{K^3 - 1} - \frac{p_o K^3(1 + a^3/2r^3)}{K^3 - 1} \tag{10.108}$$

$$\sigma_r = \frac{p_i(1 - b^3/r^3)}{K^3 - 1} - \frac{p_o K^3(1 - a^3/r^3)}{K^3 - 1} \tag{10.109}$$

If $p_i = 0$, $\sigma_r = 0$ at $r = a$, then

$$u_a = (1 - \nu)\frac{\sigma_t}{E} a \tag{10.110}$$

Conversely, if $p_o = 0$, $\sigma_r = 0$ at $r = b$, then

$$u_b = (1 - \nu)\frac{\sigma_t}{E} b \tag{10.111}$$

10.9.3 Plates

The formulas that apply for plates are based on the assumptions that the plate is flat, of uniform thickness, and of homogeneous isotropic material, thickness is not greater than one-fourth the least transverse dimension, maximum deflection is not more than one-half the thickness, all forces are normal to the plane of the plate, and the plate is nowhere stressed beyond the elastic limit. In Table 10.12 are formulas for deflection and stress for various shapes, forms of load and edge conditions. For further information see Refs. 12 and 21.

10.9.4 Trunnion

A solid shaft (Fig. 10.28) on a round or rectangular plate loaded with a bending moment is called a trunnion. The loading generally is developed from a bearing mounted on the solid shaft. For a round, simply supported plate

$$\sigma_r = \frac{\beta M}{at^2} \tag{10.112}$$

$$\theta = \frac{\gamma M}{Et^3} \tag{10.113}$$

$$\left.\begin{aligned} \beta &= 10^{(0.7634 - 1.252x)} \\ \log \gamma &= 0.248 - \pi x^{1.5} \end{aligned}\right\} 0 < x = \frac{b}{a} < 1 \tag{10.114}$$

For the fixed-end plate

$$\left.\begin{aligned} \beta &= 10^{(1 - 1.959x)} \\ \log \gamma &= 0.179 - 3.75x^{1.5} \end{aligned}\right\} 0 < x = \frac{b}{a} < 1 \tag{10.115}$$

The equations for β, γ are derived from curve fitting of data (see, for example, Refs. 2, 4th ed., and 21).

10.9.5 Socket Action

In Fig. 10.29a, summation of moments in the middle of the wall yields

$$2\left[\left(\frac{\omega''}{2}\frac{l}{2}\right)\left(\frac{2}{3}\frac{l}{2}\right)\right] = F\left(a + \frac{l}{2}\right)$$

$$\omega'' = \frac{6}{l^2}\left[F\left(a + \frac{l}{2}\right)\right] \tag{10.116}$$

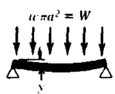
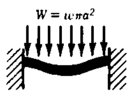
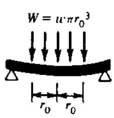
Summation of forces in the horizontal gives

$$\omega' = \frac{F}{l} \tag{10.117}$$

At B, the bearing pressure in Fig. 10.29c is

Table 10.12 Formulas for Flat Plates[a]

Notation: W = total applied load, lb; w = unit applied load, psi; t = thickness of plate, in.; σ = stress at surface of plate, psi; y = vertical deflection of plate from original position, in.; E = modulus of elasticity; m = reciprocal of ν, Poisson's ratio. q denotes any given point on the surface of plate; r denotes the distance of q from the center of a circular plate. Other dimensions and corresponding symbols are indicated on figures. Positive sign for σ indicates tension at upper surface and equal compression at lower surface; negative sign indicates reverse condition. Positive sign for y indicates upward deflection, negative sign downward deflection. Subscripts r, t, a, and b used with σ denote, respectively, radial direction, tangential direction, direction of dimension a, and direction of dimension b. All dimensions are in inches. All logarithms are to the base e ($\log_e x = 2.3026 \log_{10} x$).

Type of Load and Support	Formulas for Stress and Deflection
	Circular Flat Plates 
Outer edges supported. Uniform load over entire surface.  	At center: $\max \sigma_r = \sigma_t = \frac{-3W}{8\pi m t^2}(3m+1) \qquad \max y = -\frac{3W(m-1)(5m+1)a^2}{16\pi E m^2 t^3}$ At q: $\sigma_r = -\frac{3W}{8\pi m t^2}\left[(3m+1)\left(1-\frac{r^2}{a^2}\right)\right] \qquad \sigma_t = -\frac{3W}{8\pi m t^2}\left[(3m+1)-(m+3)\frac{r^2}{a^2}\right]$ $y = -\frac{3W(m^2-1)}{8\pi E m^2 t^3}\left[\frac{(5m+1)a^2}{2(m+1)}+\frac{r^4}{2a^2}-\frac{(3m+1)r^2}{m+1}\right]$
Outer edges fixed. Uniform load over entire surface.  	At center: $\sigma_r = \sigma_t = -\frac{3W(m+1)}{8\pi m t^2} \qquad \max y = -\frac{3W(m^2-1)a^2}{16\pi E m^2 t^3}$ At q: $\sigma_r = \frac{3W}{8\pi m t^2}\left[(3m+1)\frac{r^2}{a^2}-(m+1)\right] \qquad \sigma_t = \frac{3W}{8\pi m t^2}\left[(m+3)\frac{r^2}{a^2}-(m+1)\right]$ $y = \frac{-3W(m^2-1)}{16\pi E m^2 t^3}\left[\frac{(a^2-r^2)^2}{a^2}\right]$
Outer edges supported. Uniform load over concentric circular area of radius r_0.  	At q, $r < r_0$: $\sigma_r = -\frac{3W}{2\pi m t^2}\left[m+(m+1)\log\frac{a}{r_0}-(m-1)\frac{r_0^2}{4a^2}-(3m+1)\frac{r^2}{4r_0^2}\right]$ $\sigma_t = -\frac{3W}{2\pi m t^2}\left[m+(m+1)\log\frac{a}{r_0}-(m-1)\frac{r_0^2}{4a^2}-(m+3)\frac{r^2}{4r_0^2}\right]$ $y = -\frac{3W(m^2-1)}{16\pi E m^2 t^3}\left[4a^2-5r_0^2+\frac{r^4}{r_0^2}-(8r^2+4r_0^2)\log\frac{a}{r_0}-\frac{2(m-1)r_0^2(a^2-r^2)}{(m+1)a^2}+\frac{8m(a^2-r^2)}{m+1}\right]$ At q, $r > r_0$: $\sigma_r = -\frac{3W}{2\pi m t^2}\left[(m+1)\log\frac{a}{r}-(m-1)\frac{r_0^2}{4a^2}+(m-1)\frac{r_0^2}{4r^2}\right]$ $\sigma_t = -\frac{3W}{2\pi m t^2}\left[(m-1)+(m+1)\log\frac{a}{r}-(m-1)\frac{r_0^2}{4a^2}-(m-1)\frac{r_0^2}{4r^2}\right]$ $y = -\frac{3W(m^2-1)}{16\pi E m^2 t^3}\left[\frac{(12m+4)(a^2-r^2)}{m+1}-\frac{2(m-1)r_0^2(a^2-r^2)}{(m+1)a^2}-(8r^2+4r_0^2)\log\frac{a}{r}\right]$ At center: $\max \sigma_r = \sigma_t = -\frac{3W}{2\pi m t^2}\left[m+(m+1)\log\frac{a}{r_0}-(m-1)\frac{r_0^2}{4a^2}\right]$ $\max y = -\frac{3W(m^2-1)}{16\pi E m^2 t^3}\left[\frac{(12m+4)a^2}{m+1}-4r_0^2\log\frac{a}{r_0}-\frac{(7m+3)r_0^2}{m+1}\right]$

[a] By permission from Ref. 22.

Table 10.12 (Continued)

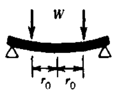
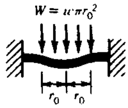
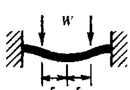
TYPE OF LOAD AND SUPPORT	FORMULAS FOR STRESS AND DEFLECTION
	CIRCULAR FLAT PLATES
Outer edges supported. Uniform load on concentric circular ring of radius r_0.  	At q, $r < r_0$: $\max \sigma_r = \sigma_t = -\frac{3W}{2\pi m t^2}\left[\frac{1}{2}(m-1) + (m+1)\log\frac{a}{r_0} - (m-1)\frac{r_0^2}{2a^2}\right]$ $y = -\frac{3W(m^2-1)}{2\pi E m^2 t^3}\left[\frac{(3m+1)(a^2-r^2)}{2(m+1)} - (r^2+r_0^2)\log\frac{a}{r_0} + (r^2-r_0^2) - \frac{(m-1)r_0^2(a^2-r^2)}{2(m+1)a^2}\right]$ At q, $r > r_0$: $\sigma_r = -\frac{3W}{2\pi m t^2}\left[(m+1)\log\frac{a}{r} + (m-1)\frac{r_0^2}{2r^2} - (m-1)\frac{r_0^2}{2a^2}\right]$ $\sigma_t = -\frac{3W}{2\pi m t^2}\left[(m-1) + (m+1)\log\frac{a}{r} - (m-1)\frac{r_0^2}{2r^2} - (m-1)\frac{r_0^2}{2a^2}\right]$ $y = -\frac{3W(m^2-1)}{2\pi E m^2 t^3}\left[\frac{(3m+1)(a^2-r^2)}{2(m+1)} - (r^2+r_0^2)\log\frac{a}{r} - \frac{(m-1)r_0^2(a^2-r^2)}{2(m+1)a^2}\right]$
Outer edges fixed. Uniform load over concentric circular area of radius r_0.  	At q, $r < r_0$: $\sigma_r = -\frac{3W}{2\pi m t^2}\left[(m+1)\log\frac{a}{r_0} + (m+1)\frac{r_0^2}{4a^2} - (3m+1)\frac{r^2}{4r_0^2}\right]$ $\sigma_t = -\frac{3W}{2\pi m t^2}\left[(m+1)\log\frac{a}{r_0} + (m+1)\frac{r_0^2}{4a^2} - (m+3)\frac{r^2}{4r_0^2}\right]$ $y = -\frac{3W(m^2-1)}{16\pi E m^2 t^3}\left[4a^2 - (8r^2+4r_0^2)\log\frac{a}{r_0} - \frac{2r^2r_0^2}{a^2} + \frac{r^4}{r_0^2} - 3r_0^2\right]$ At q, $r > r_0$: $\sigma_r = -\frac{3W}{2\pi m t^2}\left[(m+1)\log\frac{a}{r} + (m+1)\frac{r_0^2}{4a^2} + (m-1)\frac{r_0^2}{4r^2} - m\right]$ $\sigma_t = -\frac{3W}{2\pi m t^2}\left[(m+1)\log\frac{a}{r} + (m+1)\frac{r_0^2}{4a^2} - (m-1)\frac{r_0^2}{4r^2} - 1\right]$ $y = -\frac{3W(m^2-1)}{16\pi E m^2 t^3}\left[4a^2 - (8r^2+4r_0^2)\log\frac{a}{r} - \frac{2r^2r_0^2}{a^2} - 4r^2 + 2r_0^2\right]$ At center: $\sigma_r = \sigma_t = -\frac{3W}{2\pi m t^2}\left[(m+1)\log\frac{a}{r_0} + (m+1)\frac{r_0^2}{4a^2}\right] = \max \sigma_r$ when $r_0 < 0.588a$ $\max y = -\frac{3W(m^2-1)}{16\pi E m^2 t^3}\left[4a^2 - 4r_0^2\log\frac{a}{r_0} - 3r_0^2\right]$
Outer edges fixed. Uniform load on concentric circular ring of radius r_0.  	At q, $r < r_0$: $\sigma_r = \sigma_t = -\frac{3W}{4\pi m t^2}\left[(m+1)\left(2\log\frac{a}{r_0} + \frac{r_0^2}{a^2} - 1\right)\right] = \max \sigma$ when $r < 0.31a$ $y = -\frac{3W(m^2-1)}{2\pi E m^2 t^3}\left[\frac{1}{2}\left(1 + \frac{r_0^2}{a^2}\right)(a^2-r^2) - (r^2+r_0^2)\log\frac{a}{r_0} + (r^2-r_0^2)\right]$ At q, $r > r_0$: $\sigma_r = -\frac{3W}{4\pi m t^2}\left[(m+1)\left(2\log\frac{a}{r} + \frac{r_0^2}{a^2}\right) + (m-1)\frac{r_0^2}{r^2} - 2m\right]$ $\sigma_t = -\frac{3W}{4\pi m t^2}\left[(m+1)\left(2\log\frac{a}{r} + \frac{r_0^2}{a^2}\right) - (m-1)\frac{r_0^2}{r^2} - 2\right]$ $y = -\frac{3W(m^2-1)}{2\pi E m^2 t^3}\left[\frac{1}{2}\left(1 + \frac{r_0^2}{a^2}\right)(a^2-r^2) - (r^2+r_0^2)\log\frac{a}{r}\right]$ At center: $\max y = -\frac{3W(m^2-1)}{2\pi E m^2 t^3}\left[\frac{1}{2}(a^2-r_0^2) - r_0^2\log\frac{a}{r_0}\right]$

Table 10.12 (*Continued*)

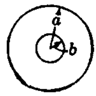
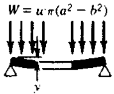
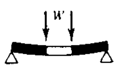
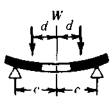
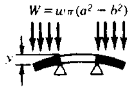
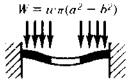
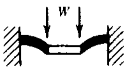
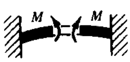
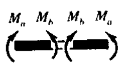
Type of Load and Support	Formulas for Stress and Deflection
	Circular Flat Plates with Concentric Circular Hole 
Outer edge supported. Uniform load over entire surface. $W = w\pi(a^2 - b^2)$ 	At inner edge: $\max \sigma = \sigma_t = -\frac{3w}{4mt^2(a^2-b^2)}\left[a^4(3m+1)+b^4(m-1)-4ma^2b^2-4(m+1)a^2b^2\log\frac{a}{b}\right]$ $\max y = -\frac{3w(m^2-1)}{2m^2Et^3}\left[\frac{a^4(5m+1)}{8(m+1)}+\frac{b^4(7m+3)}{8(m+1)}-\frac{a^2b^2(3m+1)}{2(m+1)} + \frac{a^2b^2(3m+1)}{2(m-1)}\log\frac{a}{b}-\frac{2a^2b^4(m+1)}{(a^2-b^2)(m-1)}\left(\log\frac{a}{b}\right)^2\right]$
Outer edge supported. Uniform load along inner edge. 	At inner edge: $\max \sigma = \sigma_t = -\frac{3W}{2\pi mt^2}\left[\frac{2a^2(m+1)}{a^2-b^2}\log\frac{a}{b}+(m-1)\right]$ $\max y = -\frac{3W(m^2-1)}{4\pi Em^2t^3}\left[\frac{(a^2-b^2)(3m+1)}{(m+1)}+\frac{4a^2b^2(m+1)}{(m-1)(a^2-b^2)}\left(\log\frac{a}{b}\right)^2\right]$
Supported along concentric circle near outer edge. Uniform load along concentric circle near inner edge. 	At inner edge: $\max \sigma = \sigma_t = -\frac{3W}{2\pi mt^2}\left[\frac{2a^2(m+1)}{a^2-b^2}\log\frac{c}{d}+(m-1)\frac{c^2-d^2}{a^2-b^2}\right]$
Inner edge supported. Uniform load over entire surface. $W = w\pi(a^2 - b^2)$ 	At inner edge: $\max \sigma = \sigma_t = \frac{3w}{4mt^2(a^2-b^2)}\left[4a^4(m+1)\log\frac{a}{b}+4a^2b^2+b^4(m-1)-a^4(m+3)\right]$ At outer edge: $\max y = \frac{3w(m-1)}{16Em^2t^3}\left[a^4(7m+3)+b^4(5m+1)-a^2b^2(12m+4) - \frac{4a^2b^2(3m+1)(m+1)}{(m-1)}\log\frac{a}{b}+\frac{16a^4b^2(m+1)^2}{(a^2-b^2)(m-1)}\left(\log\frac{a}{b}\right)^2\right]$
Outer edge fixed and supported. Uniform load over entire surface. $W = w\pi(a^2 - b^2)$ 	At outer edge: $\max \sigma_r = \frac{3w}{4t^2}\left[a^2-2b^2+\frac{b^4(m-1)-4b^4(m+1)\log\frac{a}{b}+a^2b^2(m+1)}{a^2(m-1)+b^2(m+1)}\right] = \nu \max \sigma$ At inner edge: $\max \sigma_t = -\frac{3w(m^2-1)}{4mt^2}\left[\frac{a^4-b^4-4a^2b^2\log\frac{a}{b}}{a^2(m-1)+b^2(m+1)}\right]$ $\max y = -\frac{3w(m^2-1)}{16m^2Et^3}\left[a^4+5b^4-6a^2b^2+8b^4\log\frac{a}{b} + \frac{\left\{[-8b^6(m+1)+4a^2b^4(3m+1)+4a^4b^2(m+1)]\log\frac{a}{b}-16a^2b^4(m+1)\left(\log\frac{a}{b}\right)^2 + 4a^2b^4-2a^4b^2(m+1)+2b^6(m-1)\right\}}{a^2(m-1)+b^2(m+1)}\right]$

Table 10.12 (*Continued*)

Type of Load and Support	Formulas for Stress and Deflection
	Circular Flat Plates with Concentric Circular Hole
Outer edge fixed and supported. Uniform load along inner edge.  	At outer edge: $\max \sigma_r = \frac{3W}{2\pi t^2}\left[1 - \frac{2mb^2 - 2b^2(m+1)\log\frac{a}{b}}{a^2(m-1)+b^2(m+1)}\right] = \max \sigma$ when $\frac{a}{b} < 2.4$ At inner edge: $\max \sigma_t = \frac{3W}{2\pi m t^2}\left[1 + \frac{ma^2(m-1) - mb^2(m+1) - 2(m^2-1)a^2\log\frac{a}{b}}{a^2(m-1)+b^2(m+1)}\right]$ $= \max \sigma$ when $\frac{a}{b} > 2.4$ $\max y = -\frac{3W(m^2-1)}{4\pi m^2 E t^3} \times$ $\left[a^2 - b^2 + \frac{2mb^2(a^2-b^2) - 8ma^2b^2\log\frac{a}{b} + 4a^2b^2(m+1)\left(\log\frac{a}{b}\right)^2}{a^2(m-1)+b^2(m+1)}\right]$
Outer edge fixed. Uniform moment along inner edge.  	At inner edge: $\max \sigma_r = \frac{6M}{t^2}$ $\max y = \frac{6M(m^2-1)}{mEt^3}\left[\frac{a^2b^2 - b^4 - 2a^2b^2\log\frac{a}{b}}{a^2(m-1)+b^2(m+1)}\right]$ At outer edge: $\sigma_r = -\frac{6M}{t^2}\left[\frac{2mb^2}{(m+1)b^2+(m-1)a^2}\right]$
Outer edge supported. Unequal uniform moments along edges.  	At q: $\sigma_r = \frac{6}{t^2(a^2-b^2)}\left[a^2M_a - b^2M_b - \frac{a^2b^2}{r^2}(M_a - M_b)\right]$ $\sigma_t = \frac{6}{t^2(a^2-b^2)}\left[a^2M_a - b^2M_b + \frac{a^2b^2}{r^2}(M_a - M_b)\right]$ From outer edge level: $y = \frac{12(m^2-1)}{mEt^3(a^2-b^2)}\left[\frac{a^2-r^2}{2}\left(\frac{a^2M_a - b^2M_b}{m+1}\right) + \log\frac{a}{r}\left(\frac{a^2b^2(M_a-M_b)}{m-1}\right)\right]$

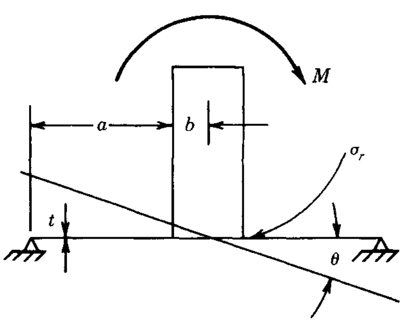


Fig. 10.28 Simply supported trunnion.

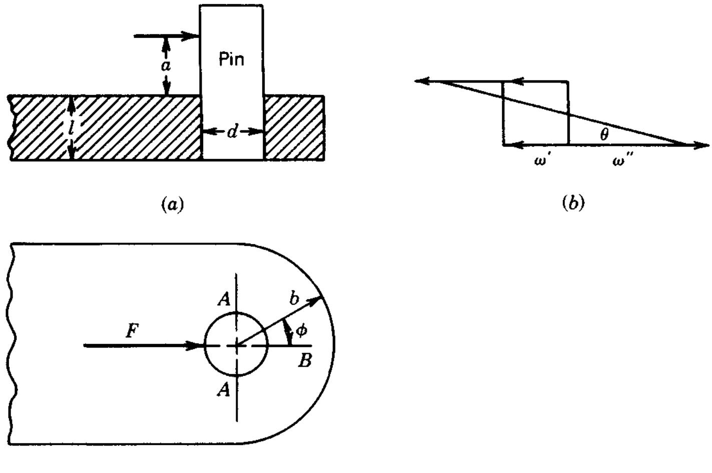


Fig. 10.29 Socket action near an edge.

$$p_i = \frac{\omega' + \omega''}{d} \text{ psi} \tag{10.118}$$

In Eq. (10.102) $p_o = 0$ and

$$\sigma_t = \frac{p_i}{R^2 - 1}\left[1 + \left(\frac{b}{d/2}\right)^2\right]$$

At A in Fig. 10.29c

$$\sigma = \frac{\phi 8F}{\pi^2 bl}$$

where $2b/d = 2, 4$ and $\phi = 4.3, 4.4$;

$$F = (\omega' + \omega'')l$$

If a pin is pressed into the frame hole, σ_t created by p_f [Eq. (10.104)] must be added. Furthermore, if the pin and frame are different metals, additional σ_t will be created by temperature changes that vary p_f.

The stress in the pin can be found from the maximum moment developed by ω' and ω'', and then calculating the bending stress.

10.10 CONTACT STRESSES

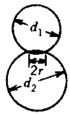
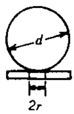
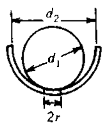
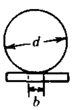
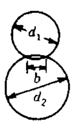
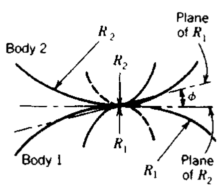
The stresses caused by the pressure between elastic bodies (Table 10.13) are of importance in connection with the design or investigation of ball and roller bearings, trunnions, expansion rollers, track stresses, gear teeth, etc.

Contact Stress Theory

H. Hertz[23] developed the mathematical theory for the surface stresses and the deformations produced by pressure between curved bodies, and the results of his analysis are supported by research. Formulas based on this theory give the maximum compressive stresses which occur at the center of the surfaces of contact, but do not consider the maximum subsurface shear stresses nor the maximum tensile stresses which occur at the boundary of the contact area. In Table 10.13 formulas are given for the elastic stress and deformation produced by bodies in contact. Numerous tests have been made to determine the bearing strength of balls and rollers, but there is difficulty in interpreting the results

Table 10.13 Areas of Contact and Pressures with Two Surfaces in Contact

Poisson's ratio = 0.3; P = load, lb; P_1 = load per in. of length, lb; E = modulus of elasticity.

Character of Surfaces	Maximum Pressure, s, at Center of Contact, psi	Radius, r, or Width, b, of Contact Area, in.
Two spheres 	$s = 0.616 \sqrt[3]{PE^2 \left(\frac{d_1 + d_2}{d_1 d_2}\right)^2}$	$r = 0.881 \sqrt[3]{\frac{P}{E} \left(\frac{d_1 d_2}{d_1 + d_2}\right)}$
Sphere and plane 	$s = 0.616 \sqrt[3]{\frac{PE^2}{d^2}}$	$r = 0.881 \sqrt[3]{\frac{Pd}{E}}$
Sphere and hollow sphere 	$s = 0.616 \sqrt[3]{PE^2 \left(\frac{d_2 - d_1}{d_1 d_2}\right)^2}$	$r = 0.881 \sqrt[3]{\frac{P}{E} \left(\frac{d_1 d_2}{d_2 - d_1}\right)}$
Cylinder and plane 	$s = 0.591 \sqrt{\frac{P_1 E}{d}}$	$b = 2.15 \sqrt{\frac{P_1 d}{E}}$
Two cylinders 	$s = 0.591 \sqrt{P_1 E \left(\frac{d_1 + d_2}{d_1 d_2}\right)}$	$b = 2.15 \sqrt{\frac{P_1}{E} \left(\frac{d_1 d_2}{d_1 + d_2}\right)}$
General case of two bodies in contact 	$s = \frac{1.5P}{\pi cd}$	$c = \alpha \sqrt[3]{\frac{P\delta}{K}}$ $d = \beta \sqrt[3]{\frac{P\delta}{K}}$ $\delta = \frac{4}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_1'} + \frac{1}{R_2'}}$ $K = \frac{8}{3} \frac{E_1 E_2}{E_2(1-\nu_1^2) + E_1(1-\nu_2^2)}$

$$\theta = \text{arc cos} \frac{1}{4} \delta \sqrt{\left\{ \left(\frac{1}{R_1} - \frac{1}{R_1'}\right)^2 + \left(\frac{1}{R_2} - \frac{1}{R_2'}\right)^2 + 2\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)\left(\frac{1}{R_2} - \frac{1}{R_2'}\right) \cos 2\phi \right\}}$$

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
α	∞	6.612	3.778	2.731	2.136	1.754	1.486	1.284	1.128	1.00
β	0	0.319	0.408	0.493	0.567	0.641	0.717	0.802	0.893	1.00

for lack of a satisfactory criterion of failure. One arbitrary criterion of failure is the amount of allowable plastic yielding. For further information on contact stresses see Refs. 2, 24, and 25.

10.11 ROTATING ELEMENTS

10.11.1 Shafts

The stress[1] in the center of a rotating shaft or solid cylinder is

$$\sigma_r = \sigma_h = \frac{3 - 2\nu}{8(1 - \nu)} \left(\frac{\gamma\omega^2}{g}\right) r_o^2 \tag{10.119}$$

$$\sigma_z = \frac{\nu\gamma\omega^2}{4g(1 - \nu)} r_o^2 \tag{10.120}$$

where ν is Poisson's ratio, ω is in rad/sec, γ is the density in lb/in.^{3}, and g is 386 in./sec^2. The limiting ω can be found by using distortion energy; however, most shafts support loads and are limited by critical speeds from torsional or bending modes of vibration. Holzer's method and Dunkerley's equation are used.

10.11.2 Disks

A rotating disk[1,9,19] of inside radius a and outside radius b has $\sigma_r = 0$ at a and b, while σ_t is

$$\sigma_{ta} = \frac{3 + \nu}{4g} \gamma\omega^2 \left(b^2 + \frac{1 - \nu}{3 + \nu} a^2\right) \tag{10.121}$$

$$\sigma_{tb} = \frac{3 + \nu}{4g} \gamma\omega^2 \left(a^2 + \frac{1 - \nu}{3 + \nu} b^2\right) \tag{10.122}$$

Substitution in Eq. (10.105) gives the outside and inside radial expansions.

The solid disk of radius b has stresses at the center

$$\sigma_t = \sigma_r = \frac{3 + \nu}{8g} \gamma\omega^2 b^2 \tag{10.123}$$

Substitution into the distortion energy [Eq. (10.22)] can give one the limiting speed.

10.11.3 Blades

Blades attached to a rotating shaft will experience a tensile force at the attachment to the shaft. These can be found from dynamics of machinery texts; however, the forces developed from a fluid driven by the blades develop more problems. The blades, if not in the plane, will develop additional forces and moments from the driving force plus vibration of the blades on the shaft.

10.12 DESIGN SOLUTION SOURCES AND GUIDELINES

Designs are composed of simple elements, as discussed here. These elements are subjected to temperature extremes, vibrations, and environmental effects that cause them to creep, buckle, yield, and corrode. Finding solutions to model these cases, as elements, can be difficult and when found the solutions are complex to follow, let alone to calculate. See Refs. 2, 21, and 26–32 Handbooks cataloging known solutions. Always cross-check with another reference. The Handbooks of Roark & Young and Blevins have been computerized using a TK solver and are distributed by UTS software. These closed form solutions would ease some of the more complicated calculations and checks finite element solutions using a computer.

10.12.1 Computers

Most computer set-ups use linear elastic solutions where the analyst supplies mechanical properties of materials such as yield and ultimate strengths and cross-sectional properties like area and area moments of inertia. When solving more complex problems, some concerns to keep in mind:

Questions to Be Asked

1. Will I know if this model buckles?
2. Can one use a non-linear stress-strain curve?
3. Is there any provision for creep and buckling?

4. How large and complex a structure can be solved? Look at a solved problem and relate it to future problems.

Things to Watch and Note

1. Press fit joints, flanges, pins, bolts, welds and bonds, and any connection interface present modeling problems. The stress analysis of a single loaded weld is not a simple task. The stress solution for a trunnion with more complexity, such as seal grooves in the plate, requires many small finite elements to converge to a closed form solution (Eq. 10.112).
2. Vibration solutions with connection interfaces can give frequency solutions with 50% error with many connections and still have 10–15% error with no connections. The computer solution appears to be always on the high side.
3. Detailed fatigue stresses on elements can be derived out of the loads by printing out the force variation.
4. Materials. The materials have good operating range[33] and limitations for spring stress relaxation at higher temperatures or lower limits can be applicable to structural members.

Nickel Alloys, Inconels and similar materials.	$-300°F \le T \le 1020°F$
300, 400, 17-4, 17-7 stainless or austenitic, martensite, and precipitation-hardening stainless steels.	$-110°F \le T \le 570°F$
Spring steels	$-5°F \le T \le 430°F$
Patented cold drawn carbon steels	$110°F \le T \le 300°F$
Copper Beryllium	$-330°F \le T \le 260°F$
Titanium Alloys	
Bronzes	$-40°F \le T \le 175°F$
Aluminum	$-300°F \le T \le 400°F$
Magnesium	$-300°F \le T \le 350°F$

The high temperatures are for the onset of creep and stress relaxation and lower mechanical properties with higher temperature. The low temperatures show higher mechanical properties but are shock-sensitive. Always examine for the mechanical properties for the temperature range and thermal expansion.[34–37] The mechanical properties at room temperature have predictable distributions with ample sample sizes, but if the temperature is varied, similar published results are not readily available.

Rubber, plastics, and elastomers have glassy transition temperatures below which the material is putty-like and above which the material is rock-like and brittle. All material mechanical properties vary a great deal due to temperature. This makes computer solutions much more complex. Testing is the final reliable check.

10.12.2 Testing

Most designs must pass some sets of vibration, environmental, and screen testing before delivery to a customer. It is at this time that design flaws show up and frequencies, stresses, and so on are verified. Some preliminary testing might help:

1. Compare impact hammer frequency test of part of or an entire system to the computer and hand calculations. The physical testing includes the boundary values sometimes difficult to simulate on a computer.
2. Spot bond optical parts to dissimilar metal structural frame, which must be hot and cold soak tested to see if the bonding fractures the optical parts. Computers cannot predict a failure of this type well.
3. Check testing of joints and seal surfaces with pressure-sensitive gaskets to see if the developed pressures are sufficient to maintain the design to proper requirements. Then use operational testing to check for thermal warping of these critical surfaces.
4. Pressurize or load brazed, welded, or soldered part to check the process and its calculations for the pressures and loads.
5. Rapid Prototyping.[38] This method could be used to check a photoelastic model by vibrating it or freezing stresses in the model from static loads. It also could define areas of high stress for a smaller grid finite element modeling. Stress coating on a regular plastic model could also point out areas of high stress.

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